Automata computation of branching laws for endomorphisms of Cuntz algebras

Katsunori Kawamura[2]
College of Science and Engineering Ritsumeikan University,
1-1-1 Noji Higashi, Kusatsu, Shiga 525-8577,Japan

Abstract

In our previous articles, we have presented a class of endomorphisms of the Cuntz algebras which are defined by polynomials of canonical generators and their conjugates. We showed the classification of some case under unitary equivalence by help of branching laws of permutative representations. In this article, we construct an automaton which is called the Mealy machine associated with the endomorphism in order to compute its branching law. We show that the branching law is obtained as outputs from the machine for the input of information of a given representation.

1 Introduction

In [8, 9], we introduced a class of endomorphisms of the Cuntz algebra $\mathcal{O}_N$ which are called permutative endomorphisms. They are given by noncommutative polynomials in canonical generators of $\mathcal{O}_N$. Such endomorphisms were motivated by an interest of the following endomorphism ρ_ν of $\mathcal{O}_3$ discovered by Noboru Nakanishi:

$$\begin{cases} \rho_\nu(s_1) \equiv s_2s_3s_1^* + s_3s_1s_2^* + s_1s_2s_3^*, \\ \rho_\nu(s_2) \equiv s_3s_2s_1^* + s_1s_3s_2^* + s_2s_1s_3^*, \\ \rho_\nu(s_3) \equiv s_1s_1s_1^* + s_2s_2s_2^* + s_3s_3s_3^* \end{cases} \tag{1.1}$$

where s_1, s_2, s_3 are canonical generators of $\mathcal{O}_3$. Because $\rho_\nu(s_1), \rho_\nu(s_2), \rho_\nu(s_3)$ satisfy the relation of canonical generators of $\mathcal{O}_3$, we can verify that ρ_ν is an endomorphism of $\mathcal{O}_3$. ρ_ν is very concrete but its property is not so clear. In Theorem 1.2 of [8], we proved that ρ_ν is irreducible but not an automorphism by using branching laws of ρ_ν with respect to permutative representations. Especially, ρ_ν is not unitarily equivalent to the canonical endomorphism of $\mathcal{O}_3$.

[2]e-mail:kawamura@kurims.kyoto-u.ac.jp.

In general, representations of C*-algebras do not have unique decomposition (up to unitary equivalence) into sums or integrals of irreducibles. However, the permutative representations of $\mathcal{O}_N$ do [1, 3, 4]. Because a representation arising from the right transformation of a permutative representation by a permutative endomorphism is also a permutative representation, their branching laws make sense. By such branching laws, permutative endomorphisms are characterized and classified effectively.

Definition 1.1. *Let $s_1, \ldots, s_N$ be canonical generators of $\mathcal{O}_N$ and $(\mathcal{H}, \pi)$ be a representation of $\mathcal{O}_N$.*

(i) *$(\mathcal{H}, \pi)$ is a permutative representation of $\mathcal{O}_N$ if there is a complete orthonormal basis $\{e_n\}_{n\in\Lambda}$ of $\mathcal{H}$ and a family $f = \{f_i\}_{i=1}^N$ of maps on Λ such that $\pi(s_i)e_n = e_{f_i(n)}$ for each $n \in \Lambda$ and $i = 1, \ldots, N$.*

(ii) *For $J = (j_i)_{i=1}^k \in \{1, \ldots, N\}^k$, $(\mathcal{H}, \pi)$ is $P(J)$ if there is a unit cyclic vector $\Omega \in \mathcal{H}$ such that $\pi(s_J)\Omega = \Omega$ and $\{\pi(s_{j_i} \cdots s_{j_k})\Omega\}_{i=1}^k$ is an orthonormal family in $\mathcal{H}$ where $s_J \equiv s_{j_1} \cdots s_{j_k}$.*

(iii) *$(\mathcal{H}, \pi)$ is a cycle if there is $J \in \{1, \ldots, N\}^k$ such that $(\mathcal{H}, \pi)$ is $P(J)$.*

For any $J \in \{1, \ldots, N\}^k$, $P(J)$ exists uniquely up to unitary equivalence. In Theorem 1.3 of [9], we showed the following:

Theorem 1.2. *Let $\mathfrak{S}_{N,l}$ be the set of all permutations on the set $\{1, \ldots, N\}^l$. For $\sigma \in \mathfrak{S}_{N,l}$, let ψ_σ be the endomorphism of $\mathcal{O}_N$ defined by*

$$\psi_\sigma(s_i) \equiv u_\sigma s_i \quad (i = 1, \ldots, N) \tag{1.2}$$

where $u_\sigma \equiv \sum_{J\in\{1,\ldots,N\}^l} s_{\sigma(J)}(s_J)^*$. If a representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ is $P(J)$ for $J \in \{1, \ldots, N\}^k$ and $\sigma \in \mathfrak{S}_{N,l}$, then there are $J_1, \ldots, J_M \in \bigcup_{m\geq 1}\{1, \ldots, N\}^m$ and subrepresentations $\pi_1, \ldots, \pi_M$ of $\pi \circ \psi_\sigma$ such that

$$\pi \circ \psi_\sigma = \pi_1 \oplus \cdots \oplus \pi_M, \tag{1.3}$$

π_i is $P(J_i)$ and $J_i \in \coprod_{n=1}^{N^{l-1}} \{1, \ldots, N\}^{nk}$ for $i = 1, \ldots, M$. Further $1 \leq M \leq N^{l-1}$.

ψ_σ in (1.2) is called the *permutative endomorphism* of $\mathcal{O}_N$ by σ. The canonical endomorphism of $\mathcal{O}_N$ and ρ_ν in (1.1) are permutative endomorphisms.

By the uniqueness of decomposition of permutative representation, the rhs in (1.3) is unique up to unitary equivalence. When $(\mathcal{H}, \pi)$ is $P(J)$ and

$\rho \in \mathrm{End}\mathcal{O}_N$, we denote $(\mathcal{H}, \pi \circ \rho)$ by $P(J) \circ \rho$ simply. Then (1.3) can be rewritten as follows:

$$P(J) \circ \psi_\sigma = P(J_1) \oplus \cdots \oplus P(J_M). \tag{1.4}$$

We call (1.4) by the *branching law* of ψ_σ with respect to $P(J)$. The branching law of ψ_σ is unique up to unitary equivalence of ψ_σ. Concrete such branching laws are already given in [8, 9] by direct computation. These branching laws are interesting subjects themselves and they are useful to classify endomorphisms effectively. On the other hand, an automaton is a typical object to consider algorithm of computation in the computer science [5, 6, 7, 10]. An automaton is a machine which changes the internal state by an input. A Mealy machine is a kind of automaton with output.

In this article, we show a better algorithm to compute branching law, that is, an algorithm to seek $J_1, \ldots, J_M$ from a given J in (1.4) by reducing the problem to a semi-Mealy machine M_σ as an input ($= J$) and outputs ($= J_1, \ldots, J_M$):

$$\underset{\textsf{Input word}}{a_J} \longrightarrow \overset{\textsf{Semi-Mealy machine}}{\boxed{\;\mathsf{M}_\sigma\;}} \longrightarrow \underset{\textsf{Output words}}{b_{J_1}, \ldots, b_{J_M}}$$

If $J = J_0^r$, that is, J is a sequence of r-times repetition of a sequence $J_0 \in \{1, \ldots, N\}^{k'}$ and $r \geq 2$, then there are $z_1, \ldots, z_r \in U(1)$ such that $P(J) = \bigoplus_{j=1}^r P(J_0) \circ \gamma_{z_j}$ where γ is the gauge action on $\mathcal{O}_N$ by Theorem 2.4 (iv) in [9]. Because $\gamma_z \circ \psi_\sigma = \psi_\sigma \circ \gamma_z$ for each z, the branching law of $P(J) \circ \psi_\sigma$ is reduced to that of $P(J_0) \circ \psi_\sigma$. Therefore it is sufficient to show the case that J is *nonperiodic*, that is, J is impossible to be written as J_0^r for $r \geq 2$. Hence we assume that J is nonperiodic.

For $\sigma \in \mathfrak{S}_{N,l}$ with $l \geq 2$ and $J \in \{1, \ldots, N\}^l$, we define $\sigma_1(J), \ldots, \sigma_l(J) \in \{1, \ldots, N\}$ by $\sigma(J) = (\sigma_1(J), \ldots, \sigma_l(J))$ and let $\sigma_{n,m}(J) \equiv (\sigma_n(J), \ldots, \sigma_m(J))$ for $1 \leq n < m \leq l$. Define $\{1, \ldots, N\}^0 \equiv \{0\}$ for convenience.

Definition 1.3. *For $\sigma \in \mathfrak{S}_{N,l}$, a data $\mathsf{M}_\sigma \equiv (Q, \Sigma, \Delta, \delta, \lambda)$ is called the semi-Mealy machine by σ if Q, Σ, Δ are finite sets,*

$$Q \equiv \{q_K : K \in \{1, \ldots, N\}^{l-1}\}, \quad \Sigma \equiv \{a_j\}_{j=1}^N, \quad \Delta \equiv \{b_j\}_{j=1}^N$$

and two maps $\delta : Q \times \Sigma^* \to Q$, $\lambda : Q \times \Sigma^* \to \Delta^*$ are defined by

$$\delta(q_K, a_i) \equiv \begin{cases} q_0 & (l = 1), \\ q_{(\sigma^{-1})_{2,l}(K,i)} & (l \geq 2), \end{cases} \qquad \lambda(q_K, a_i) \equiv \begin{cases} b_{\sigma^{-1}(i)} & (l = 1), \\ b_{(\sigma^{-1})_1(K,i)} & (l \geq 2) \end{cases}$$

for $i = 1, \ldots, N$ and $K \in \{1, \ldots, N\}^{l-1}$ where Σ^* and Δ^* are free semigroups generated by Σ and Δ, respectively.

We posteriori define $\delta(q, wa) \equiv \delta(\delta(q, w), a)$ and $\lambda(q, wa) \equiv \lambda(q, w)\lambda(\delta(q, w), a)$ for $q \in Q$, $w \in \Sigma^*$ and $a \in \Sigma$. For a given $J = (j_i)_{i=1}^k \in \{1, \ldots, N\}^k$, define $Q_J \equiv \{q \in Q : \text{there exists } n \in \mathbf{N}\, s.t.\, \delta(q, (a_J)^n) = q\}$ where $a_J \equiv a_{j_1} \cdots a_{j_k} \in \Sigma^*$ and define an equivalence relation $\sim$ in Q_J by $q \sim q^{'}$ if there is $n \in \mathbf{N}$ such that $\delta(q, (a_J)^n) = q^{'}$. Define $[q] \equiv \{q^{'} \in Q_J : q \sim q^{'}\}$. Then $[q]$ is a cyclic component of Q_J with respect to the iteration of the right action of a_J by δ. There are $p_1, \ldots, p_M \in Q_J$ such that the set Q_J of periodic points is decomposed into orbits as follows:

$$Q_J = [p_1] \sqcup \cdots \sqcup [p_M]. \tag{1.5}$$

Under these preparations, the main theorem is given as follows:

Theorem 1.4. *If J is nonperiodic, then $J_1, \ldots, J_M$ in (1.4) are obtained by*

$$b_{J_i} = \lambda(p_i, (a_J)^{r_i}) \quad (i = 1, \ldots, M)$$

where $p_1, \ldots, p_M \in Q_J$ are taken as (1.5) and $r_i \equiv \#[p_i]$ for $i = 1, \ldots, M$.

In Theorem 1.4, if $p_1^{'}, \ldots, p_M^{'}$ satisfy (1.4) and $[p_i^{'}] = [p_i]$ for each i, then the associated $J_1^{'}, \ldots, J_M^{'}$ satisfy that $P(J_i^{'}) = P(J_i)$ for each i. We show a more practical algorithm to compute branching laws by using the Mealy diagram as follows:

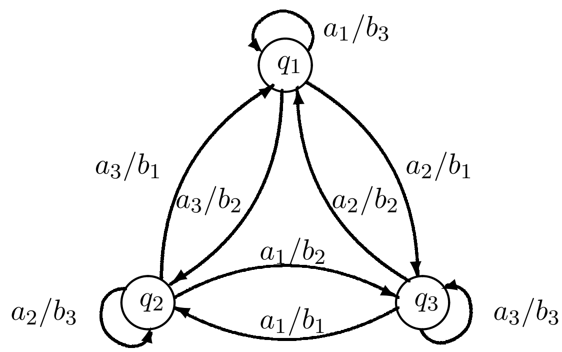
The *transition diagram (Mealy diagram)* $\mathcal{D}(\mathsf{M})$ of a semi-Mealy machine $\mathsf{M} = (Q, \Sigma, \Delta, \delta, \lambda)$ is a directed graph with labeled edges, which has a set Q of vertices and a set $E \equiv \{(q, \delta(q, a), a) \in Q \times Q \times \Sigma : q \in Q,\ a \in \Sigma\}$ of directed edges with labels. The meaning of $(q, \delta(q, a), a)$ is an edge from q to $\delta(q, a)$ with a label "$a/\lambda(q, a)$" for $a \in \Sigma$:

$$\delta(q, a) = p, \quad \lambda(q, a) = b \quad \Longleftrightarrow \quad \boxed{q} \xrightarrow{a/b} \boxed{p}$$

For ρ_ν in (1.1), we compute branching laws by the semi-Mealy machine. Define $\sigma_0 \in \mathfrak{S}_{3,2}$ by $\begin{array}{c|ccccccccc} J & 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \\ \hline \sigma_0(J) & 23 & 31 & 12 & 32 & 13 & 21 & 11 & 22 & 33 \end{array}$. Then $\rho_\nu = \psi_{\sigma_0}$ and $\mathsf{M}_{\sigma_0} = (\{q_1, q_2, q_3\}, \{a_1, a_2, a_3\},\ \{b_1, b_2, b_3\}, \delta, \lambda)$ is given as follows:

p	$\delta(p, a_1)$	$\delta(p, a_2)$	$\delta(p, a_3)$	$\lambda(p, a_1)$	$\lambda(p, a_2)$	$\lambda(p, a_3)$
q_1	q_1	q_3	q_2	b_3	b_1	b_2
q_2	q_3	q_2	q_1	b_2	b_3	b_1
q_3	q_2	q_1	q_3	b_2	b_1	b_3

From this, $\mathcal{D}(\mathsf{M}_{\sigma_0})$ is as follows:



According to Theorem 1.4, we compute branching laws for ρ_ν by $\mathcal{D}(\mathsf{M}_{\sigma_0})$. When the input word is a_1, $\delta(q_1,a_1)=q_1$, $\delta(q_2,a_1)=q_3$, $\delta(q_3,a_1)=q_2$. Therefore $Q_1=[q_1]\sqcup[q_2]$, $r_1=1$, $r_2=2$ and there are two cycles q_1 and q_2q_3 in Q with respect to a_1. From this, we have output words, $\lambda(q_1,a_1)=b_3$ and $\lambda(q_2,(a_1)^2)=b_2b_1$. Hence $P(1)\circ\rho_\nu=P(3)\oplus P(21)=P(3)\oplus P(12)$. where we use a fact that $P(j_{p(1)},\ldots,j_{p(k)})=P(j_1,\ldots,j_k)$ for each $p\in\mathbf{Z}_k$. Further the following holds:

input	cycles	outputs	branching law
a_1	q_1, q_2q_3	b_3, b_2b_1	$P(1)\circ\rho_\nu=P(3)\oplus P(12)$
a_1a_2	$q_1q_1q_3q_2q_2q_3$	$b_3b_1b_1b_3b_2b_2$	$P(12)\circ\rho_\nu=P(113223)$
$a_1a_2a_3$	$q_1q_1q_3q_3q_2q_2, q_2q_3q_1$	$b_3b_1b_3b_1b_3b_1, b_2b_2b_2$	$P(123)\circ\rho_\nu=P(131313)\oplus P(222)$
$a_1a_3a_2$	$q_1q_1q_2q_2q_3q_3, q_3q_2q_1$	$b_3b_2b_3b_2b_3b_2, b_1b_1b_1$	$P(132)\circ\rho_\nu=P(232323)\oplus P(111)$

In §2, we rewrite branching laws by branching function systems and their transformations, and we review known facts about endomorphisms. §3 is devoted to prove Theorem 1.4 by branching function systems. In §4, we show examples of Mealy diagram of the semi-Mealy machine M_σ and branching laws of ψ_σ for concrete $\sigma\in\mathfrak{S}_{N,l}$.

2 Branching function systems

In order to compute branching laws of endomorphisms, we introduce branching function systems and their transformations by permutations.

Let $\{1,\ldots,N\}^*_1\equiv\bigcup_{k\geq 1}\{1,\ldots,N\}^k$. For $J\in\{1,\ldots,N\}^*_1$, the *length* of J is defined by k when $J\in\{1,\ldots,N\}^k$. For $J_1=(j_1,\ldots,j_k), J_2=(j_1^{'},\ldots,j_l^{'})$, let $J_1\cup J_2\equiv(j_1,\ldots,j_k,j_1^{'},\ldots,j_l^{'})$. Especially, we define $(i,J)\equiv(i)\cup J$ for convenience. For J and $k\geq 2$, $J^k=J\cup\cdots\cup J$ (k-times). For

$J=(j_1,\ldots,j_k)$ and $\tau\in\mathbf{Z}_k$, define $\tau(J)\equiv(j_{\tau(1)},\ldots,j_{\tau(k)})$. For $J_1,J_2\in\{1,\ldots,N\}_1^*$, $J_1\sim J_2$ if there are $k\geq 1$ and $\tau\in\mathbf{Z}_k$ such that $J_1,J_2\in\{1,\ldots,N\}^k$ and $\tau(J_1)=J_2$. For $J_1=(j_1,\ldots,j_k), J_2=(j_1^{'},\ldots,j_k^{'})$, $J_1\prec J_2$ if $\sum_{l=1}^k(j_l^{'}-j_l)N^{k-l}\geq 0$. $J\in\{1,\ldots,N\}_1^*$ is *minimal* if $J\prec J^{'}$ for each $J^{'}\in\{1,\ldots,N\}_1^*$ such that $J\sim J^{'}$. Define $[1,\ldots,N]^*\equiv\{J\in\{1,\ldots,N\}_1^*:$ J is minimal and nonperiodic$\}$. $[1,\ldots,N]^*$ is in one-to-one correspondence with the set of all equivalence classes of nonperiodic elements in $\{1,\ldots,N\}_1^*$ with respect to the equivalence relation $\sim$.

Let Λ be an infinite set and $N\geq 2$. $f=\{f_i\}_{i=1}^N$ is a *branching function system on* Λ if f_i is an injective transformation on Λ for $i=1,\ldots,N$ such that a family of their images coincides a partition of Λ. Let $\mathrm{BFS}_N(\Lambda)$ be the set of all branching function systems on Λ. $f=\{f_i\}_{i=1}^N\in\mathrm{BFS}_N(\Lambda_1)$ and $g=\{g_i\}_{i=1}^N\in\mathrm{BFS}_N(\Lambda_2)$ are *equivalent* if there is a bijection φ from Λ_1 to Λ_2 such that $\varphi\circ f_i\circ\varphi^{-1}=g_i$ for $i=1,\ldots,N$. For $f=\{f_i\}_{i=1}^N$, we denote $f_J\equiv f_{j_1}\circ\cdots\circ f_{j_k}$ when $J=(j_1,\ldots,j_k)\in\{1,\ldots,N\}^k$ and define $f_0\equiv id$. For $x,y\in\Lambda$, $x\sim y$ (with respect to f) if there are $J_1,J_2\in\{1,\ldots,N\}^*$ and $z\in\Lambda$ such that $f_{J_1}(z)=x$ and $f_{J_2}(z)=y$. For $x\in\Lambda$, define $A_f(x)\equiv\{y\in\Lambda: x\sim y\}$. $f=\{f_i\}_{i=1}^N\in\mathrm{BFS}_N(\Lambda)$ is *cyclic* if there is an element $x\in\Lambda$ such that $\Lambda=A_f(x)$. $\{n_1,\ldots,n_k\}\subset\Lambda$ is a *cycle* of f if there is $J=(j_1,\ldots,j_k)$ such that $f_{j_1}(n_1)=n_k, f_{j_2}(n_2)=n_1,\ldots,f_{j_k}(n_k)=n_{k-1}$. f has a *cycle* if there is a cycle of f in Λ.

Let Ξ be a set and Λ_ω be an infinite set for $\omega\in\Xi$. For $f^{[\omega]}=\{f_i^{[\omega]}\}_{i=1}^N\in\mathrm{BFS}_N(\Lambda_\omega)$, f is the *direct sum* of $\{f^{[\omega]}\}_{\omega\in\Xi}$ if $f=\{f_i\}_{i=1}^N\in\mathrm{BFS}_N(\Lambda)$ for a set $\Lambda\equiv\coprod_{\omega\in\Xi}\Lambda_\omega$ which is defined by $f_i(n)\equiv f_i^{[\omega]}(n)$ when $n\in\Lambda_\omega$ for $i=1,\ldots,N$ and $\omega\in\Xi$. For $f\in\mathrm{BFS}_N(\Lambda)$, $f=\bigoplus_{\omega\in\Xi}f^{[\omega]}$ is a *decomposition* of f into a family $\{f^{[\omega]}\}_{\omega\in\Xi}$ if there is a family $\{\Lambda_\omega\}_{\omega\in\Xi}$ of subsets of Λ such that f is the direct sum of $\{f^{[\omega]}\}_{\omega\in\Xi}$. For each $f=\{f_i\}_{i=1}^N\in\mathrm{BFS}_N(\Lambda)$, there is a decomposition $\Lambda=\coprod_{\omega\in\Xi}\Lambda_\omega$ such that $\#\Lambda_\omega=\infty$, $f|_{\Lambda_\omega}\equiv\{f_i|_{\Lambda_\omega}\}_{i=1}^N\in\mathrm{BFS}_N(\Lambda_\omega)$ and $f|_{\Lambda_\omega}$ is cyclic for each $\omega\in\Xi$.

Definition 2.1. (i) *For $J\in\{1,\ldots,N\}^k$, $f\in\mathrm{BFS}_N(\Lambda)$ is $P(J)$ if f is cyclic and has a cycle $\{n_1,\ldots,n_k\}$ such that $f_J(n_k)=n_k$.*

(ii) *For $f\in\mathrm{BFS}_N(\Lambda)$ and $J\in\{1,\ldots,N\}_1^*$, g is a $P(J)$-component of f if g is a direct sum component of f and g is $P(J)$.*

For $f\in\mathrm{BFS}_N(\Lambda)$ and $\Lambda_1,\Lambda_2\subset\Lambda$, if $f|_{\Lambda_i}$ is $P(J_i)$ for $i=1,2$, then either $\Lambda_1\cap\Lambda_2=\emptyset$ or $\Lambda_1=\Lambda_2$.

Recall $\mathfrak{S}_{N,l}$ in Theorem 1.2. For $\sigma\in\mathfrak{S}_{N,l}$ and $f=\{f_i\}_{i=1}^N\in$

$\mathrm{BFS}_N(\Lambda)$, define $f^{(\sigma)}=\{f_i^{(\sigma)}\}_{i=1}^N\in \mathrm{BFS}_N(\Lambda)$ by

$$f_i^{(\sigma)}\equiv f_{\sigma(i)}\quad (l=1),\quad f_i^{(\sigma)}(f_J(n))\equiv f_{\sigma(i,J)}(n)\quad (l\geq 2) \tag{2.1}$$

for $n\in\Lambda$, $i=1,\ldots,N$ and $J\in\{1,\ldots,N\}^{l-1}$. If $\sigma\in\mathfrak{S}_N=\mathfrak{S}_{N,1}$ and $f\in \mathrm{BFS}_N(\Lambda)$ is $P(J)$, then $f^{(\sigma)}$ is $P(J_{\sigma^{-1}})$ where $J_{\sigma^{-1}}\equiv\left(\sigma^{-1}(j_1),\ldots,\sigma^{-1}(j_k)\right)$ for $J=(j_1,\ldots,j_k)$. For any $J\in\{1,\ldots,N\}_1^*$, there is $f\in\mathrm{BFS}_N(\Lambda)$ for some set Λ such that f is $P(J)$. In this case, for $\sigma\in\mathfrak{S}_{N,l}$, there is $1\leq M\leq N^{l-1}$ such that $f^{(\sigma)}$ is decomposed into a direct sum of M cycles by Lemma 2.2 in [9]. Furthermore, the length of each cycle is a multiple of that of J.

For $N\geq 2$, let $\mathcal{O}_N$ be the *Cuntz algebra* [2], that is, the C*-algebra which is universally generated by $s_1,\ldots,s_N$ satisfying $s_i^*s_j=\delta_{ij}I$ for $i,j=1,\ldots,N$ and $s_1s_1^*+\cdots+s_Ns_N^*=I$. In this article, any representation and endomorphism are assumed unital and $*$-preserving.

$(l_2(\Lambda),\pi_f)$ is the *permutative representation of* $\mathcal{O}_N$ *by* $f=\{f_i\}_{i=1}^N\in \mathrm{BFS}_N(\Lambda)$ if $\pi_f(s_i)e_n\equiv e_{f_i(n)}$ for $n\in\Lambda$ and $i=1,\ldots,N$. For $J\in\{1,\ldots,N\}_1^*$, $P(J)$ in Definition 1.1 is irreducible if and only if J is nonperiodic. For $J_1,J_2\in\{1,\ldots,N\}_1^*$, $P(J_1)\sim P(J_2)$ if and only if $J_1\sim J_2$ where $P(J_1)\sim P(J_2)$ means the unitary equivalence of two representations which satisfy the condition $P(J_1)$ and $P(J_2)$, respectively. $[1,\ldots,N]^*$ is in one-to-one correspondence with the set of equivalence classes of irreducible permutative representations of $\mathcal{O}_N$ with a cycle. If $f\in\mathrm{BFS}_N(\Lambda)$ and $g\in \mathrm{BFS}_N(\Lambda^{'})$ satisfy $f\sim g$, then $(l_2(\Lambda),\pi_f)\sim(l_2(\Lambda^{'}),\pi_g)$. If f is cyclic, then $(l_2(\Lambda),\pi_f)$ is cyclic. If f is $P(J)$, then $(l_2(\Lambda),\pi_f)$ is $P(J)$. If $\Lambda=\Lambda_1\sqcup\Lambda_2$ and $f^{(i)}\equiv f|_{\Lambda_i}\in\mathrm{BFS}_N(\Lambda_i)$ for $i=1,2$, then $(l_2(\Lambda),\pi_f)\sim(l_2(\Lambda_1),\pi_{f^{(1)}})\oplus(l_2(\Lambda_2),\pi_{f^{(2)}})$.

Let $\mathrm{End}\mathcal{A}$ be the set of all unital $*$-endomorphisms of a unital $*$-algebra $\mathcal{A}$. For $\rho\in\mathrm{End}\mathcal{A}$, ρ is *proper* if $\rho(\mathcal{A})\neq\mathcal{A}$. ρ is *irreducible* if $\rho(\mathcal{A})^{'}\cap\mathcal{A}=\mathbf{C}I$ where $\rho(\mathcal{A})^{'}\cap\mathcal{A}\equiv\{x\in\mathcal{A}:\text{for all } a\in\mathcal{A},\ \rho(a)x=x\rho(a)\}$. ρ and $\rho^{'}$ are *equivalent* if there is a unitary $u\in\mathcal{A}$ such that $\rho^{'}=\mathrm{Ad}u\circ\rho$. In this case, we denote $\rho\sim\rho^{'}$. Let $\mathrm{Rep}\mathcal{A}$ (*resp.* $\mathrm{IrrRep}\mathcal{A}$) be the set of all unital (*resp.* irreducible) $*$-representations of $\mathcal{A}$. We simply denote π for $(\mathcal{H},\pi)\in\mathrm{Rep}\mathcal{A}$. If $\rho,\rho^{'}\in\mathrm{End}\mathcal{A}$ and $\pi,\pi^{'}\in\mathrm{Rep}\mathcal{A}$ satisfy $\rho\sim\rho^{'}$ and $\pi\sim\pi^{'}$, then $\pi\circ\rho\sim\pi^{'}\circ\rho^{'}$. Assume that $\mathcal{A}$ is simple. If there is $\pi\in\mathrm{IrrRep}\mathcal{A}$ such that $\pi\circ\rho\in\mathrm{IrrRep}\mathcal{A}$, then ρ is irreducible. If there is $\pi\in\mathrm{Rep}\mathcal{A}$ such that $\pi\circ\rho\not\sim\pi\circ\rho^{'}$, then $\rho\not\sim\rho^{'}$. If there is $\pi\in\mathrm{IrrRep}\mathcal{A}$ such that $\pi\circ\rho\notin\mathrm{IrrRep}\mathcal{A}$, then ρ is proper.

For ψ_σ in (1.2), define

$$E_{N,l}\equiv\{\psi_\sigma\in\mathrm{End}\mathcal{O}_N:\sigma\in\mathfrak{S}_{N,l}\}\quad (l\geq 1). \tag{2.2}$$

If $\sigma \in \mathfrak{S}_N$, then ψ_σ is an automorphism of $\mathcal{O}_N$ which satisfies $\psi_\sigma(s_i) = s_{\sigma(i)}$ for $i = 1, \ldots, N$. Especially, if $\sigma = id$, then $\psi_{id} = id$. If $\sigma \in \mathfrak{S}_{N,2}$ is defined by $\sigma(i,j) \equiv (j,i)$ for $i,j = 1, \ldots, N$, then ψ_σ is just the canonical endomorphism of $\mathcal{O}_N$. For $\sigma \in \mathfrak{S}_{N,l}$ and $f \in \mathrm{BFS}_N(\Lambda)$, $\pi_f \circ \psi_\sigma = \pi_{f^{(\sigma)}}$ where $f^{(\sigma)}$ is in (2.1). If ρ is a permutative endomorphism and $(\mathcal{H}, \pi)$ is a permutative representation of $\mathcal{O}_N$, then $\pi \circ \rho$ is also a permutative representation.

A representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ has a $P(J)$-*component* if $(\mathcal{H}, \pi)$ has a subrepresentation $(\mathcal{H}_0, \pi|_{\mathcal{H}_0})$ which is $P(J)$. A component of a representation $P(J) \circ \rho$ of $\mathcal{O}_N$ means a subrepresentation of $(\mathcal{H}, \pi)$ which is equivalent to $P(J^{'})$ for some $J^{'}$.

For comparison of the method to find $(J_i)_{i=1}^M$ in (1.4) for a given J, we show the usual method to determine $(J_i)_{i=1}^M$ as follows: (a) Prepare a representation $(\mathcal{H}, \pi)$ which is $P(J)$. We often take $\mathcal{H} = l_2(\mathbf{N})$ and $\pi = \pi_f$ for suitable branching function system f on $\mathbf{N}$. (b) Compute $\pi(\psi_\sigma(s_i))e_n$ for each $n \in \mathbf{N}$ and $i = 1, \ldots, N$. By the proof of Lemma 2.2 in [9], we see that it is sufficient to check for $1 \leq n \leq N^{l-1}k$ when $|J| = k$. (c) Find all cycles in $\mathcal{H}$ by using results in (b). In this way, the direct computation of branching law is too much of a bother because of a great number of calculated amount when N, k, l are large.

3 Proof of Theorem 1.4

In this section, we assume that $\sigma \in \mathfrak{S}_{N,l}$, $l \geq 2$, $J = (j_i)_{i=1}^k \in \{1, \ldots, N\}^k$ and J is nonperiodic. For $r \geq 2$, extend $J = (j_i)_{i=1}^k$ as $(j_n)_{n=1}^{r \cdot k}$ by $j_{k(c-1)+i} \equiv j_i$ for each $c = 1, \ldots, r$ and $i = 1, \ldots, k$ for convenience.

Lemma 3.1. *Let $f \in \mathrm{BFS}_N(\Lambda)$ be $P(J)$, $f^{(\sigma)}$ be in (2.1) and let $M_\sigma = (Q, \Sigma, \Delta, \delta, \lambda)$ be in Definition 1.3. For $p \in Q_J$, define $r_J(p) \in \mathbf{N}$ by $r_J(p) \equiv \#[p]$.*

(i) *For $p \in Q_J$ and $\alpha \equiv r_J(p) \cdot k$, define $p_1, \ldots, p_\alpha \in Q$ and $T = (t_i)_{i=1}^\alpha \in \{1, \ldots, N\}^\alpha$ by $p_1 \equiv p$, $b_{t_1} = \lambda(p_\alpha, a_{j_\alpha})$ and*

$$p_i \equiv \delta(p_{i-1}, a_{j_{i-1}}), \quad b_{t_i} = \lambda(p_{i-1}, a_{j_{i-1}}) \quad (i = 2, \ldots, \alpha),$$

then there is $\Lambda(p) \subset \Lambda$ such that $f^{(\sigma)}|_{\Lambda(p)}$ is $P(T)$.

(ii) *In (i), define $T^{'} \in \{1, \ldots, N\}^\alpha$ by $b_{T^{'}} = \lambda(p, a_J^{r_J(p)})$. Then $f^{(\sigma)}|_{\Lambda(p)}$ is $P(T^{'})$.*

(iii) *If there is* Λ_0 *such that* $f^{(\sigma)}|_{\Lambda_0}$ *is* $P(T)$ *for* $T = (t_i)_{i=1}^{\alpha} \in \{1, \ldots, N\}^{\alpha}$, *then there is* $p \in Q_J$ *such that* Λ_0 *is equal to* $\Lambda(p)$ *in (i).*

(iv) *In (i),* $p \sim p'$ *if and only if* $\Lambda(p) = \Lambda(p')$.

(v) *Choose* $p_1, \ldots, p_M$ *as (1.5). Then the decomposition* $f^{(\sigma)} = f^{[1]} \oplus \cdots \oplus f^{[M]}$ *holds as a branching function system where* $f^{[i]} \equiv f^{(\sigma)}|_{\Lambda(p_i)}$ *for each* i.

Proof. Let $n_0 \in \Lambda$ such that $f_J(n_0) = n_0$. Because J is nonperiodic, such n_0 is unique in Λ.
(i) Let $r \equiv r_J(p)$. There is a sequence $(I_1, \ldots, I_\alpha)$ in $\{1, \ldots, N\}^{l-1}$ such that $p_i = q_{I_i}$ for each i. By definition of δ and λ and assumption,

$$\sigma(t_1, I_1) = (I_\alpha, j_\alpha), \sigma(t_2, I_2) = (I_1, j_1), \ldots, \sigma(t_\alpha, I_\alpha) = (I_{\alpha-1}, j_{\alpha-1}). \quad (3.1)$$

Define $m(p) \equiv f_{\sigma(t_1,I)}(n_0) \in \Lambda$. Then $m(p) = f_{I_\alpha}(f_{j_\alpha}(n_0))$. By this and definition of $f^{(\sigma)}$, we can verify that $f_T^{(\sigma)}(m(p)) = m(p)$. Define

$$m_\alpha \equiv m(p), \quad m_{\alpha-1} \equiv f_{t_\alpha}^{(\sigma)}(m(p)), \ldots, m_1 \equiv f_{(t_1,\ldots,t_\alpha)}^{(\sigma)}(m(p))$$

and $\Lambda(p) \equiv \{f_K^{(\sigma)}(m(p)) : K \in \{1, \ldots, N\}_1^*\}$. It is sufficient to show that $m_i \neq m_j$ when $i \neq j$. By definition,

$$m_i = f_{t_{i+1}}^{(\sigma)}(m_{i+1}) = f_{(I_i,j_i)}(f_{(j_{i+1},\ldots,j_\alpha)}(n_0)) \quad (i = 1, \ldots, \alpha-1) \quad m_\alpha = f_{t_1}^{(\sigma)}(m_1).$$

Assume that $m_i = m_{i'}$ and $c \equiv i' - i \geq 0$. This implies that $m_{\tau(i)} = m_{\tau(i')}$ for each $\tau \in \mathbf{Z}_\alpha$. From this, $(I_{\tau(i)}, j_{\tau(i)}) = (I_{\tau(i')}, j_{\tau(i')})$ and $f_{(t_{i+1},\ldots,t_\alpha)}^{(\sigma)}(m(p)) = f_{(t_{i'+1},\ldots,t_\alpha)}^{(\sigma)}(m(p))$. This implies that $f_{(I_\alpha,j_\alpha)}(n_0) = f_{(I_c,j_c)}(f_{(j_{c+1},\ldots,j_\alpha)}(n_0))$. Therefore $n_0 = f_{(j_{c+1},\ldots,j_\alpha)}(n_0)$. By the uniqueness of the cycle in Λ with respect to f, $c = k(d-1)$ for $1 \leq d \leq r$. Hence $I_{\tau(i)} = I_{\tau(i+k(d-1))}$ for each τ. Therefore $p_{\tau(i)} = q_{I_{\tau(i)}} = q_{I_{\tau(i+k(d-1))}} = p_{\tau(i+k(d-1))}$ for each τ. By the choice of r, $d = 1$ and $i = i'$. Hence the statement holds.
(ii) We see that $t_1' = t_\alpha, t_2' = t_1, \ldots, t_\alpha' = t_{\alpha-1}$. Hence $P(T) \sim P(T')$ by definition.
(iii) Fix $\tau \in \mathbf{Z}_\alpha$. Define $T' = (t_i')_{i=1}^{\alpha} \in \{1, \ldots, N\}^{\alpha}$ by

$$t_i' \equiv t_{\tau^{-1}(i)} \quad (i = 1, \ldots, \alpha). \quad (3.2)$$

Then $f^{(\sigma)}|_{\Lambda_0}$ is also $P(T')$ and there is $m_0 \in \Lambda_0$ such that $f_{T'}^{(\sigma)}(m_0) = m_0$. Define $m_\alpha \equiv m_0$ and $m_i \equiv f^{(\sigma)}(t_{i+1}, \ldots, t_\alpha)(m_0)$ for $i = 1, \ldots, \alpha - 1$.

Then $m_i \neq m_{i'}$ when $i \neq i'$. By definition of f, there are $n' \in \Lambda$, $I_0 \in \{1,\ldots,N\}^{l-1}$ and $u_0 \in \{1,\ldots,N\}$ such that $m_\alpha = f_{(I_0,u_0)}(n')$. Define a sequence $(I_i')_{i=1}^{\alpha}$ in $\{1,\ldots,N\}^{l-1}$ and $U = (u_i)_{i=1}^{\alpha} \in \{1,\ldots,N\}^{\alpha}$ by

$$I_\alpha' \equiv I_0, \quad u_\alpha \equiv u_0, \quad (I_i', u_i) \equiv \sigma(t_{i+1}', I_{i+1}') \quad (i = \alpha-1, \alpha-2, \ldots, 1).$$

By assumption, we see that $f_{(I_\alpha', u_\alpha)}(n') = f_{\sigma(t_1', I_1')}(f_U(n'))$. By definition of f, $(I_\alpha', u_\alpha) = \sigma(t_1', I_1')$ and $n' = f_U(n')$. By the uniqueness of cycle in Λ with respect to f, $U \sim J^r$. Hence there is $\tau' \in \mathbf{Z}_\alpha$ such that $j_i = u_{\tau'(i)}$ for $i = 1,\ldots,\alpha$. Here choose τ in (3.2) by $\tau \equiv \tau'$ and define $I_i \equiv I_{\tau(i)}'$ for each i. Then (3.1) holds. From this, we can verify that $p \equiv q_{I_1}$ belongs to Q_J. Define $m(p) \equiv f_{(I_\alpha, j_\alpha)}(n_0)$ as (i). Then $n_0 = f_{(j_1,\ldots,j_{\tau^{-1}(\alpha)})}(n')$ and $m_\alpha = f^{(\sigma)}_{(t_{\tau^{-1}(1)},\ldots,t_\alpha)}(m(p))$. Therefore $m_\alpha \in \Lambda(p)$. Since $m_\alpha \in \Lambda_0 \cap \Lambda(p)$, $\Lambda_0 = \Lambda(p)$.
(iv) If $p \sim p'$, then there is c such that $p' = p_{kc+1}$ in (i) and we can verify that $m(p') = f^{(\sigma)}_{(t_{1+kc},\ldots,t_\alpha)}(m(p)) \in \Lambda(p)$. Since $m(p') \in \Lambda(p') \cap \Lambda(p)$, $\Lambda(p') = \Lambda(p)$.

Assume that $\Lambda(p) = \Lambda(p')$. Let $m(p), m(p') \in \Lambda$ be in the proof of (i). Then there are $T, T' \in \{1,\ldots,N\}_1^*$ such that $f_T^{(\sigma)}(m(p)) = m(p)$ and $f_{T'}^{(\sigma)}(m(p')) = m(p')$. Then $f^{(\sigma)}|_{\Lambda(p)}$ is $P(T)$ and $f^{(\sigma)}|_{\Lambda(p')}$ is $P(T')$. Since $f^{(\sigma)}|_{\Lambda(p)} = f^{(\sigma)}|_{\Lambda(p')}$, $T' \sim T$. Assume that $T = (t_i)_{i=1}^{\alpha}$ and $T' = (t_i')_{i=1}^{\alpha}$. Let $\{m_i\}_{i=1}^{\alpha}$ be the cycle in $\Lambda(p)$ of $f^{(\sigma)}$ in (i). By the uniqueness of the cycle in $\Lambda(p)$ with respect to $f^{(\sigma)}$, $\{m_i\}_{i=1}^{\alpha}$ is also the cycle in $\Lambda(p')$ of $f^{(\sigma)}$. By the proof of (i), $m(p') \in \{m_i\}_{i=1}^{\alpha}$. Hence there is $\tau \in \mathbf{Z}_\alpha$ such that $m(p') = m_{\tau(\alpha)}$. From this, $t_i' = t_{\tau(i)}$ for $i = 1,\ldots,\alpha$. Because $T \sim T'$, $r_J(p') = r_J(p)$. Let $r \equiv r_J(p)$. Assume that $p = q_{I_1}$ and $p' = q_{I_1'}$. By definition of $m(p)$ and $m(p')$ and their relation, we see that $I_1' = I_{\tau(1)}$. Therefore $p' = q_{I_{\tau(1)}}$. By choice of p and p', $\delta(p, a_J^r) = p$ and $\delta(p', a_J^r) = p'$. Because J is nonperiodic, $\tau(i) = i + kc$ for a certain c modulo α. Therefore $p' = q_{I_{\tau(1)}} = q_{I_{1+kc}} = \delta(p, a_J^c)$. Therefore $p' \sim p$.
(v) If $i \neq j$, then $\Lambda(p_i) \neq \Lambda(p_j)$ by (iv). Hence $\Lambda(p_i) \cap \Lambda(p_j) = \emptyset$. Therefore $\Lambda(p_1) \sqcup \cdots \sqcup \Lambda(p_M) \subset \Lambda$. By (iii) and the decomposability of the branching function $f^{(\sigma)}$, $\Lambda(p_1) \sqcup \cdots \sqcup \Lambda(p_M) = \Lambda$. This implies the statement. $\square$

Proof of Theorem 1.4. Assume that $J = (j_i)_{i=1}^{k} \in \{1,\ldots,N\}^k$. When $l = 1$, $Q_J = \{q_0\}$. Let $J_{\sigma^{-1}} \equiv (\sigma^{-1}(j_1),\ldots,\sigma^{-1}(j_k))$. Then we can check that $\lambda(q_0, a_J) = b_{J_{\sigma^{-1}}}$ and $P(J) \circ \psi_\sigma = P(J_{\sigma^{-1}})$ independently. Hence the asser-

tion is verified. Assume that $l \geq 2$. By applying the correspondence between branching function systems and permutative representations, we see that the decomposition in Lemma 3.1 (v) implies that in (1.4). By definition of J_i and applying Lemma 3.1 (i), (ii) to each component in the decomposition, the statement holds. $\square$

By Theorem 1.4, it is not necessary for computation of branching law (1.4) to prepare any representation space. Further Theorem 1.4 implies the following:

Proposition 3.2. *If the Mealy diagram of M_σ has M connected components, then $P(J) \circ \psi_\sigma$ has M components of direct sum at least for each J.*

4 Examples

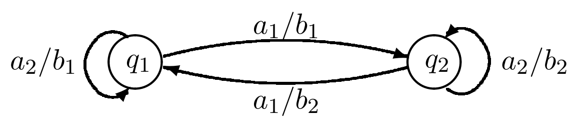
We show examples of permutative endomorphism of $\mathcal{O}_N$ and compute their branching laws by using the Mealy diagram according to Theorem 1.4. Recall $E_{N,l}$ in (2.2). Here we often denote $(j_1, \ldots, j_k)$ by $j_1 \cdots j_k$ simply.

4.1 $E_{2,2}$

In [8], we show that there are 16 equivalence classes in $E_{2,2}$ and there are 5 irreducible and proper classes $\mathcal{E}$ in them. We treat 3 elements in $\mathcal{E}$ here. For each $\sigma \in \mathfrak{S}_{2,2}$, $\mathsf{M}_\sigma = (Q, \Sigma, \Delta, \delta, \lambda)$ consists of $Q = \{q_1, q_2\}$, $\Sigma = \{a_1, a_2\}$ and $\Delta = \{b_1, b_2\}$.

Define a transposition $\sigma \in \mathfrak{S}_{2,2}$ by $\sigma(1,1) \equiv (1,2)$. Then ψ_σ and the Mealy diagram $\mathcal{D}(\mathsf{M}_\sigma)$ of M_σ are as follows:

$$\begin{cases} \psi_\sigma(s_1) \equiv s_1 s_2 s_1^* + s_1 s_1 s_2^*, \\ \psi_\sigma(s_2) \equiv s_2, \end{cases}$$



ψ_σ is irreducible and proper (Table II in [8]). We denote ψ_σ by ψ_{12} in convenience. We show several branching laws by ψ_{12}:

input	cycles	outputs	branching law
a_1	q_1q_2	b_1b_2	$P(1) \circ \psi_{12} = P(12)$
a_2	q_1, q_2	b_1, b_2	$P(2) \circ \psi_{12} = P(1) \oplus P(2)$
a_1a_2	$q_1q_2q_2q_1$	$b_1b_2b_2b_1$	$P(12) \circ \psi_{12} = P(1122)$
$a_1a_1a_2a_2$	$q_1q_2q_1q_1, q_2q_1q_2q_2$	$b_1b_2b_1b_1, b_2b_1b_2b_2$	$P(1122) \circ \psi_{12} = P(1112) \oplus P(1222)$

Focusing attention on closed paths in $\mathcal{D}(\mathsf{M}_\sigma)$, we can verify the following:

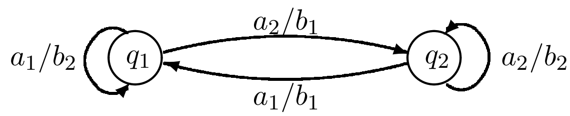
Proposition 4.1. *For each $J \in \{1,2\}_1^*$, there are J_1, J_2 or J_3 such that*

$$P(J) \circ \psi_{12} = \begin{cases} P(J_1) \oplus P(J_2) & (n_1(J) = \text{ even}), \\ \\ P(J_3) & (n_1(J) = \text{ odd}) \end{cases}$$

where $n_1(J) \equiv \sum_{l=1}^{k}(2 - j_l)$ for $J = (j_1, \dots, j_k) \in \{1,2\}^k$.

Let $\sigma \in \mathfrak{S}_{2,2}$ be a transposition defined by $\sigma(1,1) \equiv (2,1)$. Then ψ_σ, $\mathcal{D}(\mathsf{M}_\sigma)$ and branching laws of ψ_σ are given as follows:

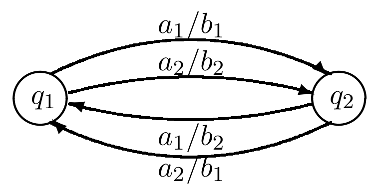
$$\begin{cases} \psi_\sigma(s_1) \equiv s_2 s_1 s_1^* + s_1 s_2 s_2^*, \\ \\ \psi_\sigma(s_2) \equiv s_1 s_1 s_1^* + s_2 s_2 s_2^*, \end{cases}$$



input	cycles	outputs	branching law
a_1	q_1	b_2	$P(1) \circ \psi_\sigma = P(2)$
a_2	q_2	b_2	$P(2) \circ \psi_\sigma = P(2)$
a_1a_2	q_2q_1	b_1b_2	$P(12) \circ \psi_\sigma = P(11)$
$a_1a_1a_2$	$q_2q_1q_1$	$b_1b_2b_1$	$P(112) \circ \psi_\sigma = P(112)$
$a_1a_2a_2$	$q_2q_1q_2$	$b_1b_1b_2$	$P(122) \circ \psi_\sigma = P(112)$

Let $\sigma \in \mathfrak{S}_{2,2}$ be defined by $\sigma(1,1) \equiv (2,2)$, $\sigma(1,2) \equiv (1,1)$, $\sigma(2,1) \equiv (2,1)$, $\sigma(2,2) \equiv (1,2)$. Then ψ_σ, $\mathcal{D}(\mathsf{M}_\sigma)$ and branching laws are as follows:

$$\begin{cases} \psi_\sigma(s_1) \equiv s_2 s_2 s_1^* + s_1 s_1 s_2^*, \\ \\ \psi_\sigma(s_2) \equiv s_2 s_1 s_1^* + s_1 s_2 s_2^*, \end{cases}$$



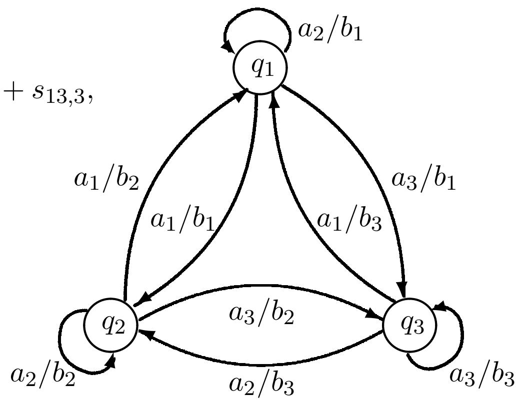
input	cycles	outputs	branching law
a_1	q_1q_2	b_1b_2	$P(1) \circ \psi_\sigma = P(12)$
a_2	q_1q_2	b_2b_1	$P(2) \circ \psi_\sigma = P(12)$
a_1a_2	q_1q_2, q_2q_1	b_1b_1, b_2b_2	$P(12) \circ \psi_\sigma = P(11) \oplus P(22)$

4.2 $E_{3,2}$

Note that $\#E_{2,2} = 2^2! = 24$ and $\#E_{3,2} = 3^2! \sim 3.6 \times 10^5$. Hence it is difficult to classify every element in $E_{3,2}$ by computing its branching laws in comparison with the case $E_{2,2}$. We see that $\mathsf{M}_\sigma = (\{q_1,q_2,q_3\}, \{a_1,a_2,a_3\}, \{b_1,b_2,b_3\}, \delta, \lambda)$ for each $\sigma \in \mathfrak{S}_{3,2}$. ρ_ν in (1.1) belongs to $E_{3,2}$.

Let $\sigma \in \mathfrak{S}_{3,2}$ be a transposition by $\sigma(1,1) \equiv (1,2)$. Then ψ_σ, $\mathcal{D}(\mathsf{M}_\sigma)$ and branching laws are as follows:

$$\begin{cases} \psi_\sigma(s_1) \equiv & s_{12,1} + s_{11,2} + s_{13,3}, \\ \psi_\sigma(s_2) \equiv & s_2, \\ \psi_\sigma(s_3) \equiv & s_3, \end{cases}$$



input	cycles	outputs	branching law
a_1	q_1q_2	b_1b_2	$P(1)\circ\psi_\sigma = P(12)$
a_2	q_1, q_2	b_1, b_2	$P(2)\circ\psi_\sigma = P(1)\oplus P(2)$
a_3	q_3	b_3	$P(3)\circ\psi_\sigma = P(3)$

where $s_{ij,k} \equiv s_i s_j s_k^*$. From this, we see that ψ_σ^n is proper and irreducible for each $n \geq 1$, and ψ_σ and ρ_ν are not equivalent.

4.3 $E_{4,2}$

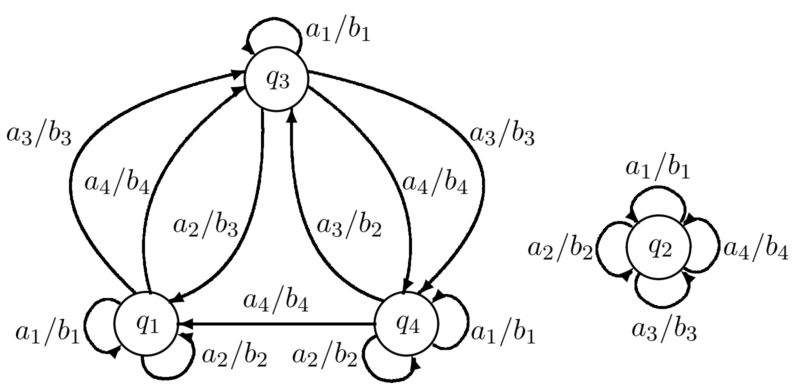
Define $\sigma \in \mathfrak{S}_{4,2}$ by

J	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
$\sigma(J)$	11	21	31	41	12	22	43	42	32	23	13	33	44	24	14	34

Then ψ_σ and $\mathcal{D}(\mathsf{M}_\sigma)$ are as follows:

$$\psi_\sigma(s_1) \equiv s_{11,1} + s_{21,2} + s_{31,3} + s_{41,4}, \quad \psi_\sigma(s_2) \equiv s_{12,1} + s_{22,2} + s_{43,3} + s_{42,4},$$

$$\psi_\sigma(s_3) \equiv s_{32,1} + s_{23,2} + s_{13,3} + s_{33,4}, \quad \psi_\sigma(s_4) \equiv s_{44,1} + s_{24,2} + s_{14,3} + s_{34,4},$$



When $J=(1)$, $\delta(q_i,a_1)=q_i$ and $\lambda(q_i,a_1)=b_1$ for each $i=1,2,3,4$. Therefore $P(1)\circ\psi_\sigma=P(1)\oplus P(1)\oplus P(1)\oplus P(1)$. In the same way, we have

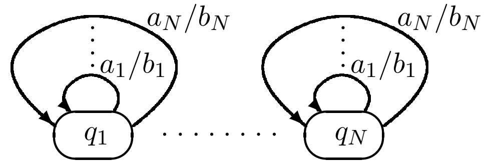
$$P(2)\circ\psi_\sigma=P(2)\oplus P(2)\oplus P(2),\quad P(4)\circ\psi_\sigma=P(4)\oplus P(444).$$

This is an example of Proposition 3.2.

4.4 Canonical endomorphism

The Mealy diagram associated with the canonical endomorphism ρ of $\mathcal{O}_N$ (see §2) is given as follows:

$$\rho(x)\equiv s_1xs_1^*+\cdots+s_Nxs_N^*,$$

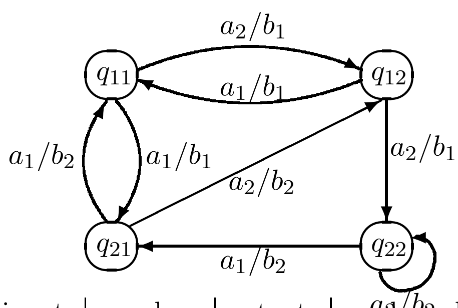


In this case, there is no transition among different states. We see that $P(J)\circ\rho=P(J)^{\oplus N}$ for each $J\in\{1,\ldots,N\}_1^*$ where $P(J)^{\oplus N}$ is the direct sum of N copies of $P(J)$. In general, $\pi\circ\rho=\pi^{\oplus N}$ for any representation π of $\mathcal{O}_N$.

4.5 $E_{2,3}$

Let $\sigma\in\mathfrak{S}_{2,3}$ be a transposition by $\sigma(1,1,1)\equiv(1,2,1)$. Then $\psi_\sigma\in E_{2,3}$, $\mathcal{D}(\mathsf{M}_\sigma)$ and branching laws are as follows:

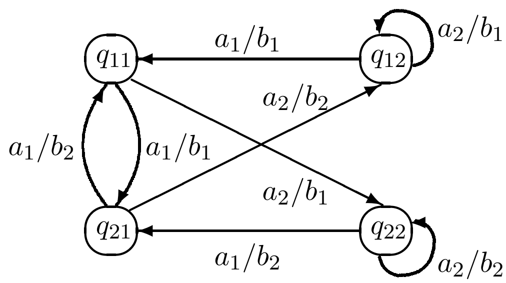
$$\begin{cases}\psi_\sigma(s_1)\equiv s_{121}s_{11}^*+s_{112}s_{12}^*+s_{111}s_{21}^*+s_{122}s_{22}^*,\\ \\ \psi_\sigma(s_2)\equiv s_2,\end{cases}$$



input	cycles	outputs	branching law
a_1	$q_{11}q_{21}$	b_1b_2	$P(1)\circ\psi_\sigma=P(12)$
a_2	q_{22}	b_2	$P(2)\circ\psi_\sigma=P(2)$
a_1a_2	$q_{12}q_{11}$	b_1b_1	$P(12)\circ\psi_\sigma=P(11)$
$a_1a_1a_2$	$q_{12}q_{11}q_{21}$	$b_1b_1b_2$	$P(112)\circ\psi_\sigma=P(112)$

We see that ψ_σ^n is irreducible and proper for each $n \geq 1$.

Let $\sigma \in \mathfrak{S}_{2,3}$ be defined by the product $\sigma = \sigma^{'} \circ \sigma^{''}$ of two transpositions $\sigma^{'}$ and $\sigma^{''}$ defined by $\sigma^{'}(1,1,1) \equiv (1,2,1)$ and $\sigma^{''}(1,1,2) \equiv (1,2,2)$, respectively. In this case $\psi_\sigma = \psi_{12} \in E_{2,2}$ in §4.1. $\mathcal{D}(\mathsf{M}_\sigma)$ is as follows:

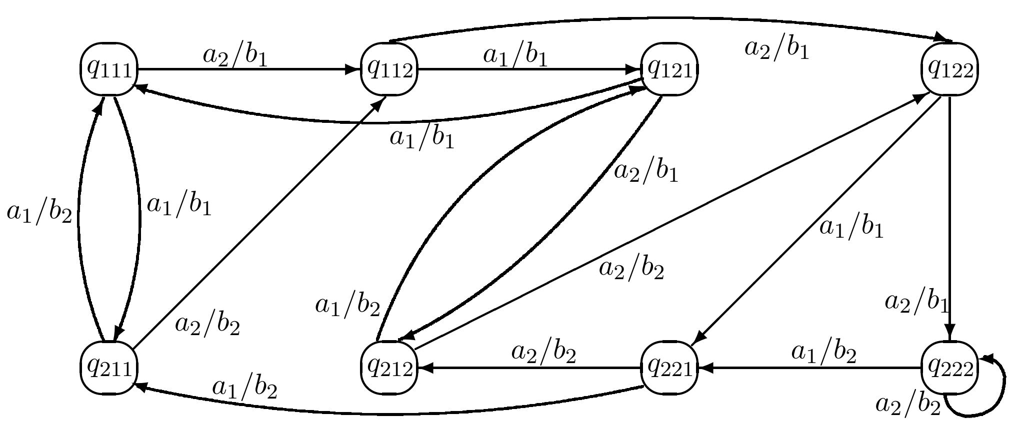


We can verify that branching laws of ψ_σ coincide with those of ψ_{12}.

4.6 $E_{2,4}$

Define a transposition $\sigma \in \mathfrak{S}_{2,4}$ by $\sigma(1,1,1,1) \equiv (1,2,1,1)$. Then $\psi_\sigma \in E_{2,4}$, $\mathcal{D}(\mathsf{M}_\sigma)$ and branching laws are given as follows:

$$\psi_\sigma(s_1) \equiv s_{1211}s_{111}^* + s_{1112}s_{112}^* + s_{112}s_{12}^* + s_{1111}s_{211}^* + s_{1212}s_{212}^* + s_{122}s_{22}^*, \quad \psi_\sigma(s_2) \equiv s_2,$$



input	cycles	outputs	branching law
a_1	$q_{111}q_{211}$	b_1b_2	$P(1) \circ \psi_\sigma = P(12)$
a_2	q_{222}	b_2	$P(2) \circ \psi_\sigma = P(2)$
a_1a_2	$q_{212}q_{121}$	b_2b_1	$P(12) \circ \psi_\sigma = P(12)$
$a_1a_1a_2$	$q_{112}q_{121}q_{111}$	$b_1b_1b_1$	$P(112) \circ \psi_\sigma = P(111)$

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